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ON ONE METHOD OF SOLVING NONSTATIONARY HEAT-CONDUCTION PROBLEMS FOR MULTILAYER STRUCTURES

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As applied to the solution of the heat-conduction problem for a two-layer structure, the Fourier method is used jointly with the orthogonal Bubnov–Galerkin method. An important feature is the introduction of additional boundary conditions, the need for which is explained by the appearance of an additional parameter μ *after the separation of the variables in the input differential equation. The additional boundary conditions are derived from the basic differential equation by differentiating it at the boundary points.*

The present paper describes a problem-definition procedure that does not require determination of the functions exactly satisfying the equations obtained after separation of the variables in the input differential equation. The basic idea of the proposed approach is as follows. The solution for the Sturm–Liouville equation obtained after the separation of the variables is constructed as a series containing coordinate functions (algebraic or trigonometric) with unknown coefficients determined from the basic and additional boundary conditions by solving the system of algebraic linear equations. To find the eigenvalues, an integral of the weighted residual of the (Sturm–Liouville) differential equation is computed. From this, to determine the eigenvalues, an algebraic polynomial whose degree depends on the number of terms of the sequence of the solution used is obtained [1, 2].

Consider the application of this method to the solution of the heat-conduction problem for a two-layer infinitely stretched plate under boundary conditions of the third kind in the following mathematical formulation:

$$\frac{\partial t_i(\eta, \tau)}{\partial \tau} = a_i \frac{\partial^2 t_i(\eta, \tau)}{\partial \eta^2} \quad (\eta_{i-1} \le \eta < \eta_i\,, \quad i = 1, 2\,, \quad \eta_0 = 0\,, \quad \eta_2 = \delta)\,, \tag{1}$$

$$t_i(\eta, 0) = t_{0i}\,, \tag{2}$$

$$\partial t_i(0, \tau)/\partial \eta = 0\,, \tag{3}$$

$$t_1(\eta_1, \tau) = t_2(\eta_1, \tau)\,, \tag{4}$$

$$\lambda_1 \partial t_1(\eta_1, \tau)/\partial \eta = \lambda_2 \partial t_2(\eta_1, \tau)/\partial \eta\,, \tag{5}$$

$$\lambda_2 \partial t_2(\eta_2, \tau)/\partial \eta = \alpha\,[t(\eta_2, \tau) - t_{\mathrm{m}}]\,. \tag{6}$$

We introduce the following dimensionless variables and parameters: $x = \eta/\delta$; $\mathrm{Fo} = a\tau/\delta^2$; $\mathrm{Bi} = \alpha\delta/\lambda$; $\Theta_i(x, \mathrm{Fo}) = (t_i - t_{\mathrm{m}})/(t_{0i} - t_{\mathrm{m}})$; a is the least of the thermal diffusivities a_i (i = 1, 2).

In view of the notations used, problem (1)–(6) takes on the form

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$$\frac{\partial \Theta_i (x, \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{a_i}{a} \frac{\partial^2 \Theta_i (x, \mathrm{Fo})}{\partial x^2} \quad (\mathrm{Fo} > 0 , \quad x_{i-1} \le x < x_i , \quad i = 1, 2 , \quad x_0 = 0 , \quad x_2 = 1) , \tag{7}$$

$$\Theta_i (x, 0) = 1 , \tag{8}$$

$$\partial \Theta_i (0, \mathrm{Fo}) / \partial x = 0 , \tag{9}$$

$$\Theta_1 (x_1, \mathrm{Fo}) = \Theta_2 (x_1, \mathrm{Fo}) , \tag{10}$$

$$\lambda_1 \partial \Theta_1 (x_1, \mathrm{Fo}) / \partial x = \lambda_2 \partial \Theta_2 (x_1, \mathrm{Fo}) / \partial x , \tag{11}$$

$$\partial \Theta_2 (1, \mathrm{Fo}) / \partial x + \mathrm{Bi}\, \Theta_2 (1, \mathrm{Fo}) = 0 . \tag{12}$$

In accordance with the method of separation of variables, the solution of problem (7)–(12) is found in the form

$$\Theta_i (x, \mathrm{Fo}) = \varphi_i (\mathrm{Fo})\, \Psi_i (x) . \tag{13}$$

Substituting (13) into (7), we find the following two ordinary differential equations:

$$\varphi_i^{\mathrm{I}} (\mathrm{Fo}) + \mu \varphi_i (\mathrm{Fo}) = 0 , \tag{14}$$

$$\frac{a_i}{a} \Psi_i^{\mathrm{II}} (x) + \mu \Psi_i (x) = 0 , \tag{15}$$

here $\mu = \nu^2$.

The solution of Eq. (14) is known:

$$\varphi_{ni} (\mathrm{Fo}) = A_n \exp (- \mu_n \mathrm{Fo}) , \tag{16}$$

where A_n denotes the unknown coefficients.

For Eq. (15), the following boundary conditions and conjugation conditions are used:

$$\Psi_1^{\mathrm{I}} (0) = 0 ; \tag{17}$$

$$\Psi_1 (x_1) = \Psi_2 (x_1) ; \tag{18}$$

$$\lambda_1 \Psi_1^{\mathrm{I}} (x_1) = \lambda_2 \Psi_2^{\mathrm{I}} (x_1) ; \tag{19}$$

$$\Psi_2^{\mathrm{I}} (1) + \mathrm{Bi}\, \Psi_2 (1) = 0 . \tag{20}$$

The solution of Eq. (15) is taken in the form

$$\Psi_{ni}(x) = \sum_{k=0}^{n} C_{ki} N_{ki}(x) \quad (i = 1, 2) , \tag{21}$$

where C_{ki} stands for the unknown coefficients; $N_{ki}(x)$ denotes the coordinate functions.

As the coordinate functions for the first and second layers, we use the functions

$$N_{k1}(x) = x^k \quad (k = 0, 2, 4, 6, ...) , \tag{22}$$

$$N_{k2}(x) = x^k \quad (k = 0, 1, 3, 5, ...) . \tag{23}$$

The unknown coefficients C_{ki} ($k = \overline{0, n}$, $i = 1, 2$) are determined from the boundary conditions (17)–(20). Since the number of coefficients C_{ki} can be arbitrarily large, and the boundary conditions and the conjugation conditions are only four, it is necessary to introduce additional boundary conditions. They can be obtained from Eq. (15) and its derivatives of different orders at the points $x = 0$ and $x = 1$. As the first additional conditions, we take

$$\Psi_1(0) = \text{const} = 1 , \tag{24}$$

which follows from the boundary condition (17). The other additional boundary conditions (at the point $x = 0$) will be of the form

$$\Psi_1^{\mathrm{II}}(0) = -\mu a_2/a_1 , \tag{25}$$

$$\Psi_1^{\mathrm{III}}(0) = 0 , \tag{26}$$

$$\Psi_1^{\mathrm{IV}}(0) = \mu^2 a_2/a_1 , \tag{27}$$

$$\Psi_1^{\mathrm{V}}(0) = 0 , \tag{28}$$

$$\Psi_1^{\mathrm{VI}}(0) = -\mu^3 a_2/a_1 , \tag{29}$$

$$\Psi_1^{\mathrm{VII}}(0) = 0 , \tag{30}$$

$$\Psi_1^{\mathrm{VIII}}(0) = \mu^4 a_2/a_1 , \tag{31}$$

$$\Psi_1^{\mathrm{IX}}(0) = 0 , \tag{32}$$

$$\Psi_1^{\mathrm{X}}(0) = -\mu^5 a_2/a_1 , \dots . \tag{33}$$

To obtain an additional boundary condition at the point $x = 1$, we differentiate Eq. (12) with respect to Fo:

$$\frac{\partial^2 \Theta_2(1, \mathrm{Fo})}{\partial x \partial \mathrm{Fo}} + \mathrm{Bi} \frac{\partial \Theta_2(1, \mathrm{Fo})}{\partial \mathrm{Fo}} = 0 . \tag{34}$$

Differentiate Eq. (7) with respect to x and write the obtained relation for the point $x = 1$:

$$\frac{\partial^2 \Theta_2 (1, \mathrm{Fo})}{\partial x \partial \mathrm{Fo}} = \frac{a_i}{a} \frac{\partial^3 \Theta_2 (1, \mathrm{Fo})}{\partial x^3} . \tag{35}$$

Rewrite relation (34) in view of (7):

$$\frac{\partial^2 \Theta_2 (1, \mathrm{Fo})}{\partial x \partial \mathrm{Fo}} = -\, \mathrm{Bi} \frac{a_i}{a} \frac{\partial^2 \Theta (1, \mathrm{Fo})}{\partial x^2} . \tag{36}$$

Comparing (35) and (36), we obtain

$$\frac{\partial^3 \Theta_2 (1, \mathrm{Fo})}{\partial x^3} + \mathrm{Bi} \frac{\partial^2 \Theta_2 (1, \mathrm{Fo})}{\partial x^2} = 0 . \tag{37}$$

From (37) we get the following additional boundary condition:

$$\Psi_2^{\mathrm{III}} (1) + \mathrm{Bi}\, \Psi_2^{\mathrm{II}} (1) = 0 . \tag{38}$$

To find the coefficients C_{ki}, substitute (21) into (17)–(20), (24)–(27), and (38). In so doing, according to (21), for the first layer we restrict ourselves to five terms of the series and for the second layer — to four terms. As a result, we will have nine algebraic linear equations (according to the number of basic and additional boundary conditions) with nine unknown C_{k1} ($k = \overline{0, 4}$) and C_{k2} ($k = \overline{0, 3}$).

Analysis of this system of equations permits the conclusion that only one unknown enters into each of the five equations containing the unknown C_{k1} (the equations are separated), which can easily be found:

$$C_{01} = 1 ; \quad C_{11} = 0 ; \quad C_{21} = -\frac{1}{2} \mu \frac{a_2}{a_1} ; \quad C_{31} = 0 ; \quad C_{41} = \frac{1}{24} \mu^2 \frac{a_2}{a_1} .$$

To determine the coefficients C_{k2} ($k = \overline{0, 3}$), it is necessary to solve four algebraic linear equations composed of the boundary conditions (18)–(20), (38).

Find the solution of problem (7)–(12) for the following input data [3]: $\eta_1 = 0.002$ m, $\eta_2 = 0.006$ m, $a_1 = 12.5 \cdot 10^{-6}$ m^2/sec, $a_2 = 6 \cdot 10^{-6}$ m^2/sec, $\lambda_1 = 45.24$ W/(m·K), $\lambda_2 = 16.24$ W/(m·K), Bi = 2, and $a = a_2 = 6 \cdot 10^{-6}$ m^2/sec.

Once the coefficients C_{ki} have been determined, the integral of the weighted residual of Eq. (15) is written as

$$\int_0^{x_1} \left(\frac{a_1}{a} \sum_{k=0}^{4} C_{k1} \frac{\partial^2 N_{k1}}{\partial x^2} + \mu \sum_{k=0}^{4} C_{k1} N_{k1} \right) dx + \int_{x_1}^{1} \left(\frac{a_2}{a} \sum_{k=0}^{3} C_{k2} \frac{\partial^2 N_{k2}}{\partial x^2} + \mu \sum_{k=0}^{3} C_{k2} N_{k2} \right) dx = 0 . \tag{39}$$

Determining the integrals in relation (39), with respect to the parameter μ we obtain the algebraic polynomial

$$-\, 8.02234 \cdot 10^{-2} \mu^2 + 1.1942 \cdot 10^{-3} \mu^3 - 1.21949 + 1.25019 \mu = 0 . \tag{40}$$

Its roots are

$$\mu_1 = 1.044343 , \quad \mu_2 = 22.316056 , \quad \mu_3 = 43.816549 . \tag{41}$$

The eigenfunctions corresponding to each eigenvalue are found from (21). Relation (13), in view of (16) and (21), for each eigenvalue will be of the form

$$\Theta_{ki} (x, \mathrm{Fo}) = A_k \Psi_{ki} (x, \mu_k) \exp (-\, \mu_k \mathrm{Fo}) \quad (k = \overline{1, 3} , \quad i = 1, 2) . \tag{42}$$

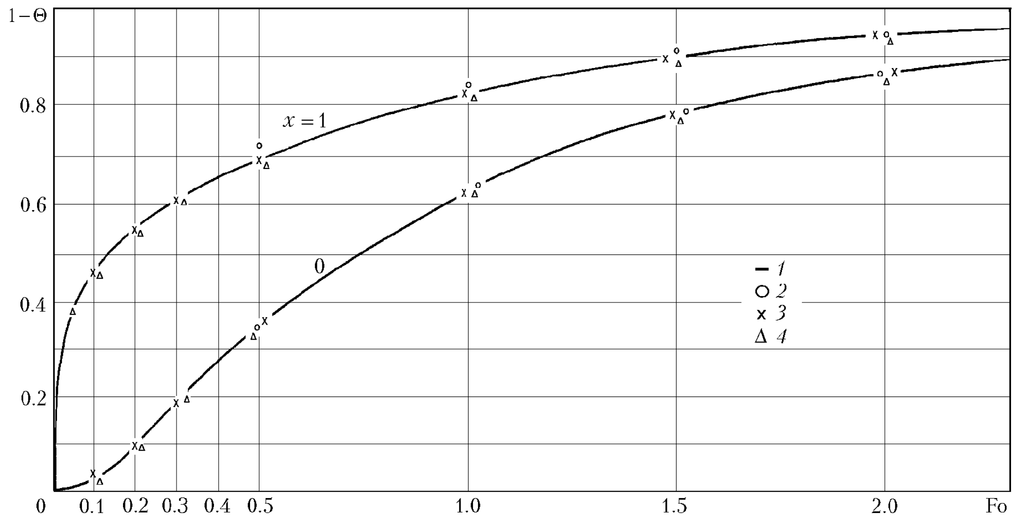

Fig. 1. Temperature distribution in the two-layer plate: 1) results of the calculations obtained by the marching method; 2) data of [3]; 3) results of the calculation obtained by formula (43) (third approximation); 4) results of the calculation obtained by formula of (5.58) [4] (sixth approximation with coordinate functions (5.20), (5.21)).

Each particular solution of the form (42) exactly satisfies the boundary conditions (9), (12) and the conjugation conditions (10), (11) and approximately (depending on the number of approximations — the number of eigenvalues found from polynomial (40)) satisfies Eq. (7). However, not a single one of these particular solutions, including their sum

$$\Theta_i\,(x, \mathrm{Fo}) = \sum_{k=1}^{3} A_k \Psi_{ki}\,(x, \mu_k)\,\exp\,(-\,\mu_k\,\mathrm{Fo}) \quad (i = 1, 2)\,, \tag{43}$$

satisfies the initial condition (8). For Eq. (43) to satisfy the initial condition, we set up its residual and require that the residual be orthogonal to each eigenfunction $\Psi_{ki}(x, \mu)$, i.e.,

$$\int_0^{x_1} \left[\sum_{k=1}^{3} A_k \Psi_{k1}\,(x, \mu_k) - 1 \right] \Psi_{j1}\,(x, \mu_j)\,dx + \int_{x_1}^{1} \left[\sum_{k=1}^{3} A_k \Psi_{k2}\,(x, \mu_k) - 1 \right] \Psi_{j2}\,(x, \mu_j)\,dx = 0 \quad (j = 1, 2, 3)\,. \tag{44}$$

Determining the integrals in (44), with respect to the unknown coefficients A_k (k = 1, 2, 3) we obtain a system of three algebraic linear equations. From its solution we find

$$A_1 = 1.119281\,, \quad A_2 = -\,0.500714\,, \quad A_3 = 0.404640\,.$$

Having determined A_k, we find the solution of problem (7)–(12) in closed form from (43).

The results of the calculations by formula (43), as compared to the data of [3], and the results of the calculations by the finite-difference method (marching method) and the method of [4] are given in Fig. 1.

Analysis of the results obtained permits the conclusion on a fair agreement between the dimensionless temperatures determined by all of the above methods.

For the number of layers $m > 2$, the coordinate functions of the third layer can be taken to be the same as the functions of the first layer, i.e., $N_{k3} = x^k$ (k = 0, 2, 4, 6, ...), those of the fourth layer — the same as the functions of the second layer, and so on. It should be noted that the conditions of linear independence of the coordinate functions between individual layers will be satisfied.

NOTATION

a_i, thermal diffusivity; t_{0i}, initial temperature; t_m, medium temperature; t_i, temperature of the ith layer; α, heat-transfer coefficient; $\delta = \delta_1 + \delta_2$, thickness of the two-layer system; δ_1, δ_2, layer thickness; η, coordinate; λ_i, heat-conductivity coefficient of the ith layer; τ, time.

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